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CONCEPTS AND PROCEDURES USED TO DETERMINE CERTAIN SEA WAVE CHARACTERISTICS

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16. Abstract

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CONCEPTS AND PROCEDURES USED TO DETERMINE

CERTAIN SEA WAVE CHARACTERISTICS

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SUMMARY

A technique and its application are presented by which wave parameters critical to a spacecraft water impact load analysis may be determined. Supporting references are cited for the applicability of a statistical approach to the problem of defining a distribution of the relevant sea surface parameters. These parameters include wave slope, wave direction, and the wave speed relative to the windspeed (wave age). This technique is based on a fully developed sea condition (windspeed: 9.5 to 29.0 knots) and is therefore directly governed by the local windspeed. A correction factor is introduced for windspeeds at which the sea is not likely to be fully developed.

A spectral equation and data used in its derivation were used to construct wave age and wave slope curves. By applying these curves, equations for the various sea wave characteristics were derived. Generalized instructions for the use of these equations and distributions are included. These procedures were used in evaluating the water impact structural capability of the Apollo command module.

INTRODUCTION

This report documents the concept used in determining wave characteristics and describes the procedures used in isolating the sea wave parameters which are significant in analysis of the impact loads encountered by a spacecraft landing in open water. The development of an accurate description of the oscillations of the sea surface and the effect of these oscillations upon spacecraft landing are problems in statistical analysis. Fortunately, a substantial amount of work on this subject has been done.

According to Cartwright (ref. 1), "The disordered oscillations of the sea ... are definable only in terms of probabilities." Also, "There have been several attempts to define a statistical model of sea waves ... they all possess the unique feature of

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arriving at essentially the same result: a random, moving surface defined by a 'stationary Gaussian process.'" (A random, statistically invariant process is one in which the variables are normally distributed, and the probability of any given event is constant over a long period of time and over a large area.) "The fact that this result and various statistical properties derived from it agree fairly well with most observations justifies its adoption until serious evidence of its failure is produced." Such a process is completely determined by its spectral function; this process will be the foundation of this report.

As an aid to the reader, where necessary the original units of measure have been converted to the equivalent value in the Système International d'Unités (SI). The SI units are written first, and the original units are written parenthetically thereafter.

SYMBOLS

$A^2(\mu)$ Neumann spectral function, J-sec/N

$A^2(\mu, \theta)$ energy wave spectra (two-dimensional), J-sec/N-rad

$B = 2g^2/v^2$

b   point intercept

C   constant in the Neumann equation

$F(\mu, \theta)$ directional part of the spectra

g   gravitational acceleration

$\vec{i}, \vec{j}, \vec{k}$ arbitrary unit vectors

K   wave age

l, m, n   direction cosines

M   wave slope correction factor

n   slope in the point-intercept equation

$S^2(\mu, \theta)$ spectral slope function

s   standardized variable

T   wave period, sec

V   windspeed, knots (except where noted)

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V_w	horizontal wave speed, knots (except where noted)
v	windspeed, m/sec
X, Y, Z	standard right-hand orthogonal coordinate system
x, y	point-intercept equation coordinates
$\vec{1}_{ud}, \vec{1}_c, \vec{1}_{NP}$	unit vectors in the wave tangent plane
γ	resultant wave slope, included angle between the wave tangent plane and the local horizontal, deg
γ_c	crosswind wave slope, angle between Y_H and the projection in the wave tangent plane, deg
γ_{ud}	upwind-downwind wave slope, angle between X_H and the projection in the wave tangent plane, deg
γ'	standard deviation, rad
δ	upper integration limit
ϵ	lower integration limit
θ	direction toward which wave is traveling when measured in a counter-clockwise direction with respect to the positive X-axis
μ	angular frequency, rad/sec, 2π/wave period
σ^2	variance of slope
ϕ	angle between wind direction and direction of wave motion projected in the horizontal plane, deg

Subscripts:

c	crosswind
H	horizontal reference axis system
NP	normal to wave tangent plane
TP	wave tangent plane reference axis system
ud	upwind-downwind

DISCUSSION

Basic Concept

In the early 1950's, Neumann (refs. 2 and 3) found that it was possible to derive a semiempirical expression for the characteristic frequency spectrum of wind waves, which for a "fully developed" sea condition, depends only on the windspeed. This function — a model of the sea surface — has been observed to approximate reality over the windspeed range discussed in this paper. Although the Neumann spectra provided no information about varying directions of wave travel, Pierson, as a result of the Stereo Wave Observation Project (SWOP) (Ref. 4), contributed to this aspect of the problem. By using the SWOP results and the Neumann one-dimensional gravity wave spectra, the two-dimensional spectra can be defined as

$$A^2(\mu, \theta) = A^2(\mu)\, F(\mu, \theta) \tag{1}$$

where

$$A^2(\mu) = \frac{\pi}{2} \frac{C e^{-2g^2 / \mu^2 v^2}}{\mu^6} \tag{2}$$

is the Neumann spectral function (ref. 5). In equation (2), μ is the angular frequency (2π/wave period) of the wave motion, g is gravitational acceleration, v is windspeed, and C is a constant (3.05×10^4 cm^2/sec^5). In the term $F(\mu, \theta)$, which represents the directional part of the spectra, the angle θ represents the directional deviation of wind and wave energy propagation.

Pierson (ref. 5) expresses the upwind-downwind component of the variance of sea surface slope as

$$\sigma_{ud}^2 = \frac{1}{2} \int_0^\infty \int_{-\pi/2}^{\pi/2} S_{ud}^2(\mu, \theta)\, d\theta\ d\mu = 1.19 \times 10^{-3} v \tag{3}$$

and the crosswind component as

$$\sigma_c^2 = \frac{1}{2} \int_0^\infty \int_{-\pi/2}^{\pi/2} S_c^2(\mu, \theta)\, d\theta\ d\mu = 0.40 \times 10^{-3} v \tag{4}$$

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where $S^2(\mu, \theta) = S_{ud}^2(\mu, \theta) + S_c^2(\mu, \theta)$ is the spectral slope function. The variance of slope, without regard to direction, can therefore be written as

$$\sigma^2 = \sigma_{ud}^2 + \sigma_c^2 = 1.19 \times 10^{-3}v + 0.40 \times 10^{-3}v \tag{5}$$

or

$$\sigma^2 = \frac{1}{2}\int_0^\infty S^2(\mu)d\mu = 1.59 \times 10^{-3}v \tag{6}$$

But from the empirical SWOP data (ref. 4), it is found that

$$\sigma_{ud}^2 = \int_0^\infty \int_{-\pi/2}^{\pi/2} A^2(\mu, \theta)\frac{\mu^4}{g^2}\cos^2\theta \; d\theta \; d\mu \approx (0.625)\int_0^\infty S^2(\mu)d\mu \tag{7}$$

and

$$\sigma_c^2 = \int_0^\infty \int_{-\pi/2}^{\pi/2} A^2(\mu, \theta)\frac{\mu^4}{g^2}\sin^2\theta \; d\theta \; d\mu \approx (0.375)\int_0^\infty S^2(\mu)d\mu \tag{8}$$

The angle θ limits carry the intuitive idea that the wave energy is confined to a sector of $\pm 90°$.

Inasmuch as the scale of interest does not require consideration of the entire range of wave frequencies (zero to infinity), the contributions of waves presenting impact surfaces, small compared to spacecraft dimensions, should be eliminated by introducing as an upper limit to the integrals a value corresponding to a wave period of 1.6 seconds. In classical wave theory, this limit yields a wavelength of approximately 3.9 meters (13 feet). Pierson, Neumann, and James (ref. 6) suggest that actual sea waves have lengths somewhat shorter than those predicted by classical theory. Furthermore, assignment of a meaningful dimension to wavelength is difficult in a short-crested sea. Therefore, the smallest elements that the model retains are those waves with a half length of approximately 1.22 to 1.83 meters (4 to 6 feet). As a lower limit (to which the integrals are relatively insensitive), the frequency at which the phase velocity of the waves is twice the wind velocity $(g/2V_w)$ is selected. Only a minute contribution to the variance results from waves of lower frequency than this.

The Neumann spectral function predicts a variance of slope that increases linearly with windspeed — observations support this. Consequently, evaluations of the right sides of equations (7) and (8) for two values of windspeed will yield equations for the standard deviation (positive square root of the variance) of the slope components γ_{ud}' and γ_c' as a function of windspeed. The resulting expressions are

$$\gamma_{ud}' = 0.79\left(0.808 \times 10^{-3}V - 0.00581\right)^{1/2} \text{ radians} \tag{9}$$

and

$$\gamma_c' = 0.612\left(0.808 \times 10^{-3}V - 0.00581\right)^{1/2} \text{ radians} \tag{10}$$

The derivation of equations (9) and (10) is presented in the appendix.

Under the Gaussian assumption, the slope distribution is then uniquely defined for any value of windspeed, provided that the sea is in an essentially fully developed condition. The reason for basing the approach on a fully developed sea condition is meteorological in nature: Windspeeds of 10 to 30 knots are frequent in the areas of concern, and the corresponding fetch and duration may quite often be sufficient to produce a sea condition that is essentially saturated in its spectral components, except possibly for the very long period waves. Admittedly, this is less likely to be the case for 30 knots than, for example, 20 knots. To compensate for probable limited duration or fetch (or both) at the higher windspeeds that might be encountered, distributions for a correction factor M are provided in figure 1.

Presented in figure 2 are the normal sea surface slope distributions for the components in the direction of the wind and at right angles to the wind for windspeeds of 9.5, 19.5, and 29.0 knots, respectively. These instantaneous, local slope components are a statistical property of the entire sea surface. Average trough-to-crest "sawtooth" slopes are not represented. For example, an ocean wave with a "sawtooth" slope of $10°$ may, at any instant of time, exhibit that value over only a very small portion of its area and in various directions. Comparison of the resultant distributions with observations may be made by reference to Cox and Munk (refs. 7 to 9) and Longuet-Higgins (ref. 10). Agreement is close for the applicable cases. Pierson (ref. 11) and Neumann and Pierson (refs. 2 and 3) provide a more general comparison.
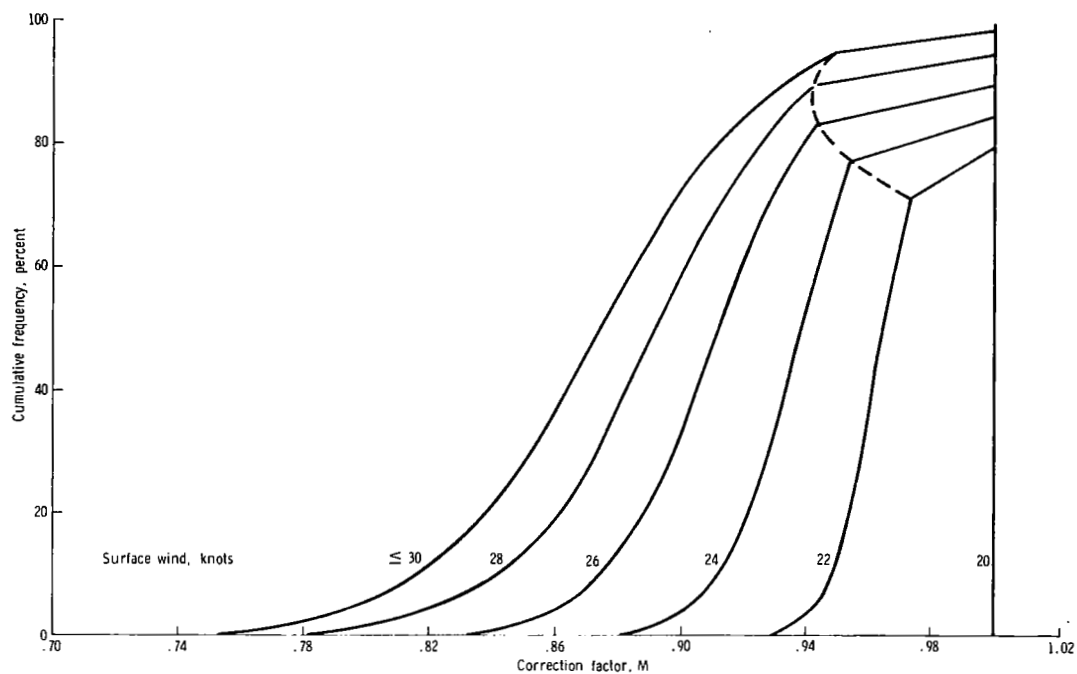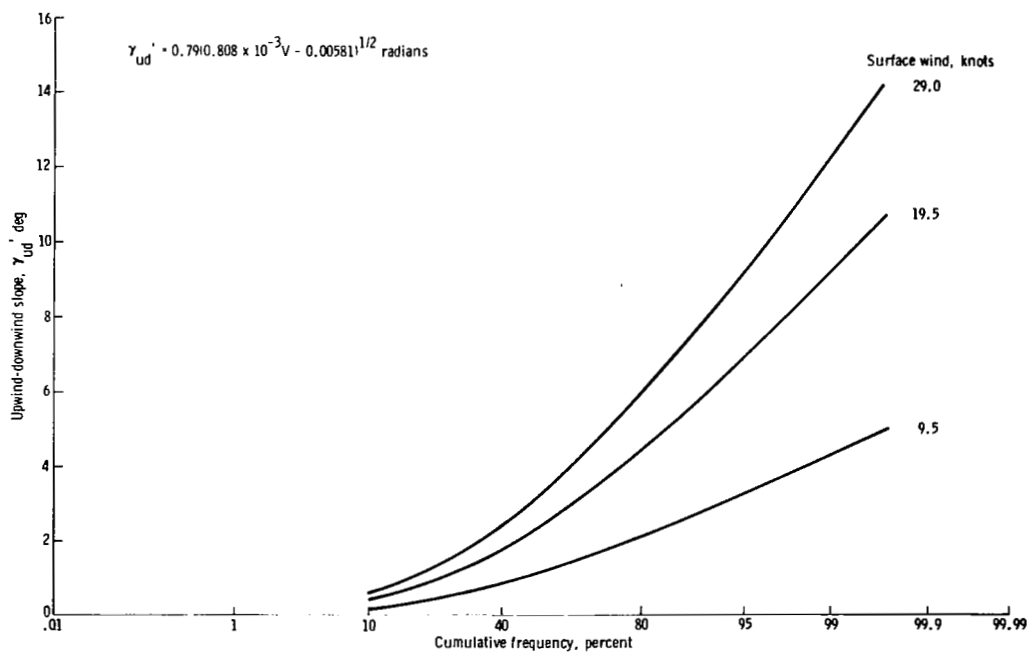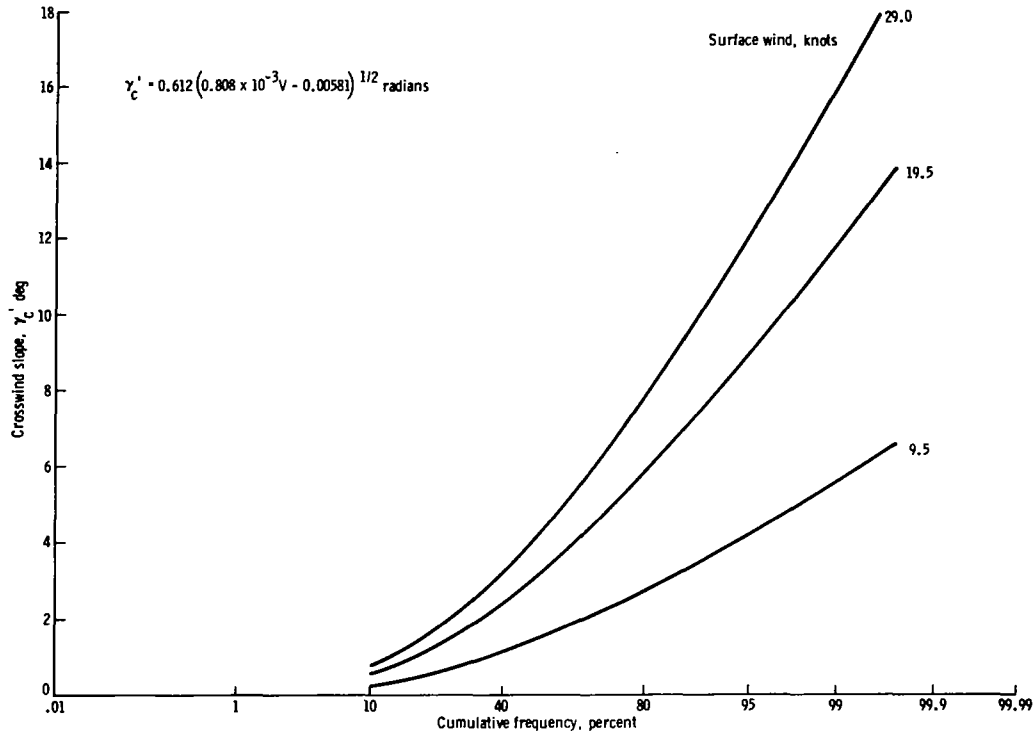
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Figure 1. - Correction factor M to compensate for less than fully developed seas.

(a) Upwind-downwind slope.

Figure 2. - Cumulative frequency distribution of the sea surface slope for various surface winds.

$$\gamma_c' = 0.612 \left(0.808 \times 10^{-3}V - 0.00581\right)^{1/2} \text{ radians}$$

(b) Crosswind slope.

Figure 2. - Concluded.

Development of Procedures

To generate sea slopes with a normal distribution, a normally distributed random variable with a mean equal to zero and variance equal to one is generated in figure 3. From this figure, two values are randomly and independently selected (s_1 and s_2) and multiplied by the standard deviations of γ_{ud}' and γ_c', respectively; the values that result are the slopes. By randomly and independently choosing a set of correction factors appropriate to the values of V selected from figure 4 (an arbitrary ocean surface wind cumulative frequency distribution) and applying those values to the slopes, the resultant upwind-downwind and crosswind slopes are determined. Finally, the resultant slopes can be expressed as

$$\gamma = \tan^{-1}\sqrt{\tan^2\gamma_{ud} + \tan^2\gamma_c} \tag{11}$$

The derivation for equation (11) is presented in the appendix.
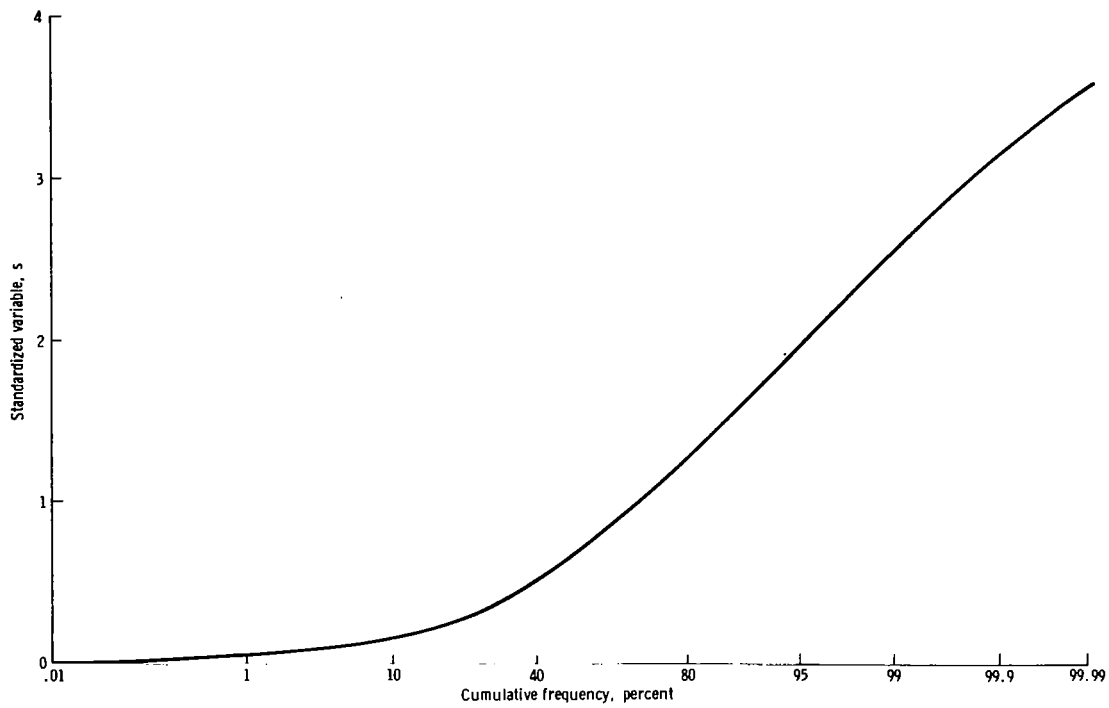
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Figure 3. – Cumulative frequency distribution for a variable with a mean of zero and a variance equal to one.
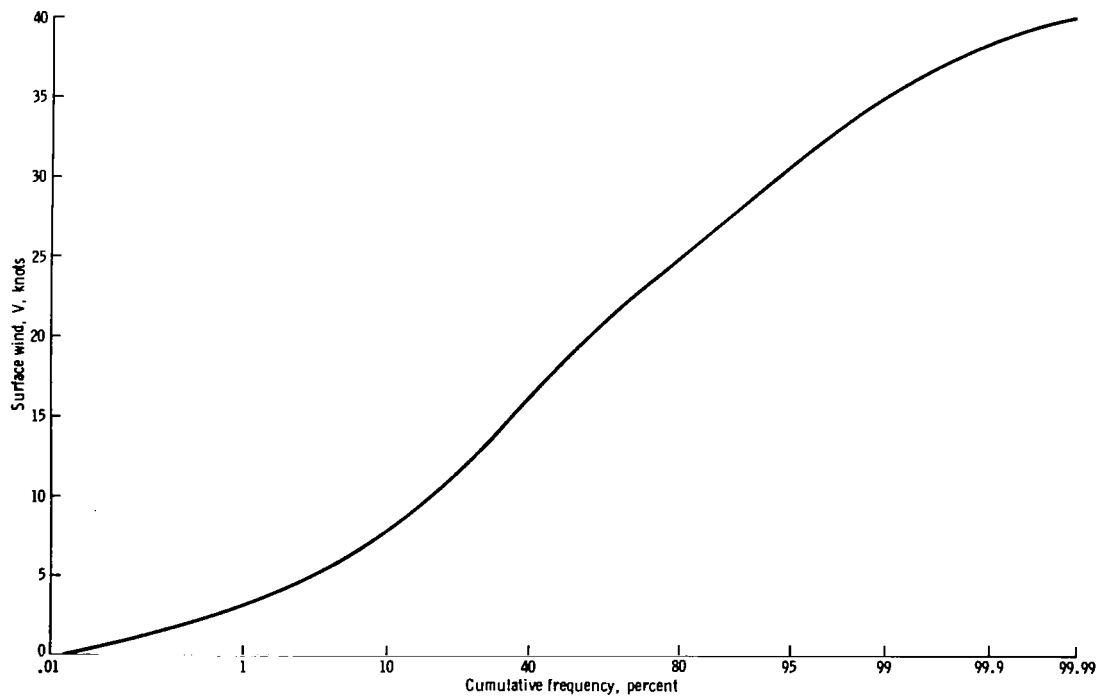

Figure 4. – Cumulative frequency distribution for a proposed set of surface winds.

The horizontal velocity of a wave can be determined from the characteristic wave age (ratio of wave speed to windspeed) K. Consequently, if the windspeed and wave age representing the selected sea state are known, the wave speed can be determined. Presented in figure 5 are the cumulative frequency distributions of wave age for surface winds. These curves are applicable to a fully developed sea at the indicated windspeed, and result from empirical distribution functions of wave periods (ref. 6). Waves that make a relatively insignificant contribution to the dominating pattern are not represented in this discussion. For high windspeeds (≥30 knots), particularly in a developing condition, some wave components below the lowest age indicated in figure 5 may be expected. These curves must therefore be regarded as deceptive in this area. This shortcoming is a property of the data from which the curves were derived. These data concern primarily the visibly significant waves at a fixed point as a function of time. No directional correction is necessary.

Inasmuch as wave age and wave steepness are interrelated (ref. 12), the distribution of wave age for the various wind velocities is stratified into three groups. The range of each group is indirectly representative of the magnitude of the upwind-downwind wave slope. Because shorter, younger waves are steeper than older components of the spectrum, slope values from the steepest one-third suggest a low wave age between
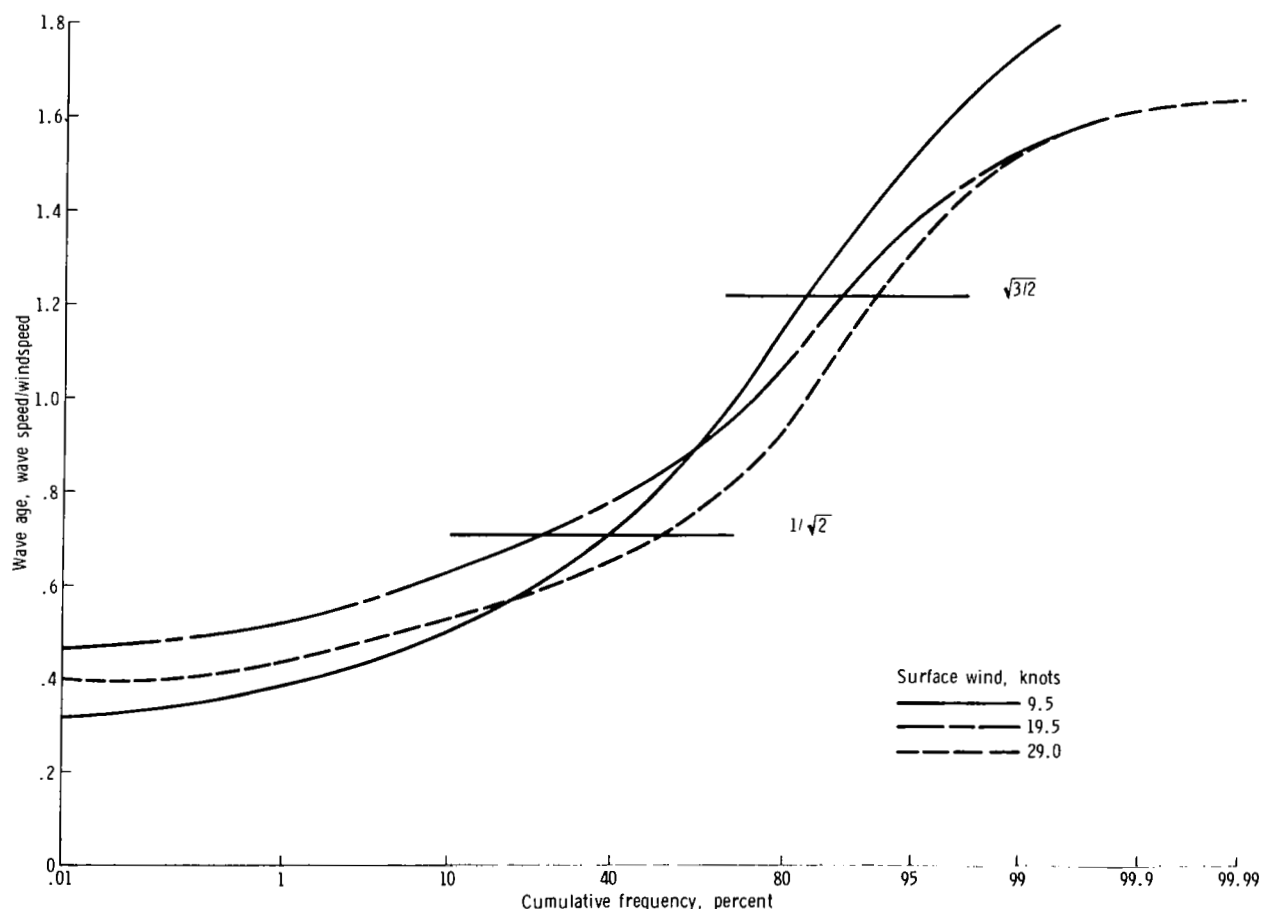

Figure 5. - Cumulative frequency distribution of wave age for various surface winds.

4.85/V and $1/\sqrt{2}$. These values of wave age correspond respectively to high-frequency cut-off and the maximums in the slope spectra. Similarly, if the wave slope is selected from the middle one-third of its range, the wave age is located between $1/\sqrt{2}$ and $\sqrt{3/2}$. These values of wave age correspond respectively to the maximums in the slope spectra and the maximums in the energy spectra. Finally, if the wave slope is in the lowest one-third of its range, the wave age is between $\sqrt{3/2}$ and 2.0. These waves contribute little to the slope spectra. Therefore, based on the percentile (which is a function of the standard deviation) of the upwind-downwind slope component, the wave age is randomly chosen between the boundaries of one of the three groups. After the wave age is selected, the horizontal wave speed is calculated by multiplying the wave age by the windspeed.

$$V_w = VK \tag{12}$$

Barber and Tucker (ref. 13) indicate that knowledge of the directional properties of sea waves is rather scanty. The wind-wave direction correlation is a complicated function that involves wave frequency and the wind field. Consequently, a simplified distribution (fig. 6) was chosen as a proposed compromise that is considered suitable for the purpose of this report. Because the older, longer wave components are more likely to run relatively straight downwind, although the younger components show greater variability, the angular deviation of the wave direction from the wind direction ϕ is associated with wave age. This decision is somewhat subjective, but is not without foundation.

SEA MODEL PROCEDURES

The sea model parameters that were defined at the NASA Manned Spacecraft Center (MSC) are as follows:

1. Upwind-downwind slope, γ_{ud}

2. Crosswind slope, γ_c

3. Horizontal wave velocity, V_w
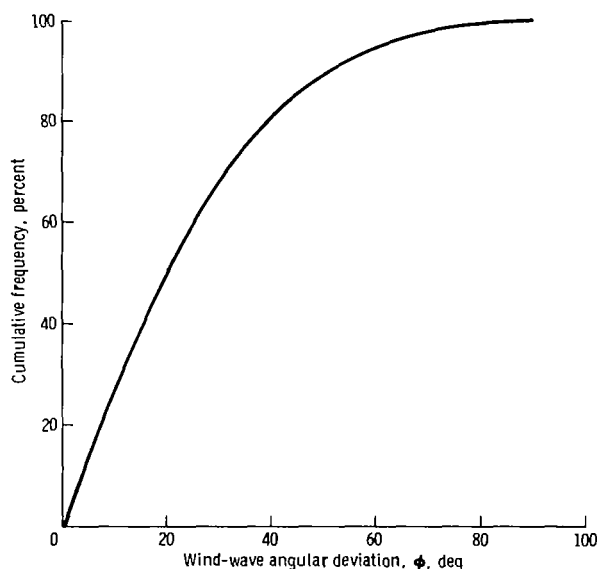
4. Wave direction, ϕ

Figure 6. – Cumulative frequency distribution for the wind-wave correlation.

The following procedures are used to determine the MSC sea model parameters:

1. By using figure 4, randomly select a percentage between 0 and 100 percent from the cumulative frequency axis and obtain the surface wind V value from the velocity axis.

2. Calculate the standard deviations for the wave slopes by using the surface wind V value obtained in step 1 and the equations (9) and (10) or data from figure 2.

3. From figure 1, randomly select two values from the cumulative frequency axis, and obtain the corresponding values of the correction factors M_1 and M_2 from the curve (interpolated if necessary) appropriate to the value of V obtained in step 1.

4. From figure 3, randomly select two values from the cumulative frequency axis, and obtain the corresponding values of the standarized variables s_1 and s_2.

5. Calculate γ_{ud} and γ_c by using the following equations.

$$\gamma_{ud} = \gamma_{ud}' M_1 s_1 (57.3) \text{ degrees} \tag{13}$$

$$\gamma_c = \gamma_c' M_2 s_2 (57.3) \text{ degrees} \tag{14}$$

6. Calculate the resultant slope γ, which represents the included angle between a normal to the surface (defined by γ_{ud} and γ_c) and the local horizontal, by using equation (11).

7. Based on the randomly selected cumulative frequency percentage used in obtaining s_1 in step 4 and on the data in figure 5, determine the wave age (wave speed/windspeed) K by one of the following procedures:

a. If cumulative frequency percentage > 67 percent, randomly select a wave age value from the segment of the appropriate curve (interpolated if necessary) that lies between $4.85/V$ and $1/\sqrt{2}$.

b. If 33 percent ≤ cumulative frequency percentage ≤ 67 percent, randomly select a wave age value from the segment of the appropriate curve (interpolated if necessary) that lies between $1/\sqrt{2}$ and $\sqrt{3/2}$.

c. If cumulative frequency percentage < 33 percent, randomly select a wave age value from the segment of the appropriate curve (interpolated if necessary) that lies between $\sqrt{3/2}$ and 1.8.

8. By using the randomly selected value for wave age K in step 7, determine the horizontal wave speed by equation (12).

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9. The randomly selected value for wave age K used in step 8, has a corresponding cumulative frequency percentage, which is obtained from figure 5. This cumulative frequency percentage and the data shown in figure 6 allow the wave direction (angular deviation of the wave direction from the wind direction ϕ) to be determined by one of the following methods:

a. If cumulative frequency percentage > 88 percent, randomly select a value for the wave direction ϕ from the ϕ-axis between 0 and 68 percent on the cumulative frequency axis.

b. If 36 percent $\leq$ cumulative frequency percentage $\leq$ 88 percent, randomly select a value for the wave direction ϕ from the ϕ-axis between 0 and 86 percent on the cumulative frequency axis.

c. If cumulative frequency percentage < 33 percent, randomly select a value for the wave direction ϕ from the ϕ-axis between 0 and 100 percent on the cumulative frequency axis.

CONCLUDING REMARKS

The purpose of this report is to document the concepts and the procedures developed for determining wave characteristics required to perform a spacecraft water impact load analysis. These parameters include wave slope, wave direction, and wave speed relative to windspeed.

A spectral equation is used to construct slope and wave age distribution curves. The slope distributions require evaluation of integrals whose normal limits are too broad for the problem. Consequently, suitable limits were chosen and references were cited for comparison of the results with applicable observations. Also, a simplified directional distribution was constructed, and a correction factor for the higher windspeeds when the sea is not likely to be in a fully developed condition was introduced.

By using distributions such as those developed in this report, a procedure was established that would provide the sea surface parameters necessary to perform a spacecraft water impact analysis. The approach taken in establishing this procedure is somewhat subjective; consequently, some inconsistencies may exist. However, the distributions that were provided (taken jointly as directed or separately) should yield reasonably predictable conditions for the fully developed sea at a given windspeed above 7.19 knots.

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National Aeronautics and Space Administration
Houston, Texas, June 7, 1972
914-50-11-09-72

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APPENDIX

EQUATION DEVELOPMENT

Figure A-1 represents the relationship between a plane denoting the local horizontal axis system X_H, Y_H, Z_H and a plane tangent to the maximum wave slope X_{TP}, Y_{TP}, Z_{TP}. The local horizontal axis system is defined by a standard right-hand orthogonal coordinate system where the X_H axis lies in the plane (positive in the direction of the wind), and Z_H is normal to the plane (positive toward the geodetic center of earth). Similarly, the tangent plane is defined by the X_{TP} and Y_{TP} axes, where X_{TP} is normal to the line formed by the intersection of this plane with the local horizontal plane, and the X_{TP} axis is positive in the upslope direction. The Z_{TP} axis is normal to the tangent plane and positive downwind. The Y_{TP} axis completes a standard right-hand orthogonal coordinate system. The angles between the tangent plane and X_H and Y_H axes in the horizontal plane denote γ_{ud} and γ_c, respectively. Likewise, the angle between the tangent plane and the X_{TP} axis, projected into the horizontal plane, represents the resultant wave slope γ.
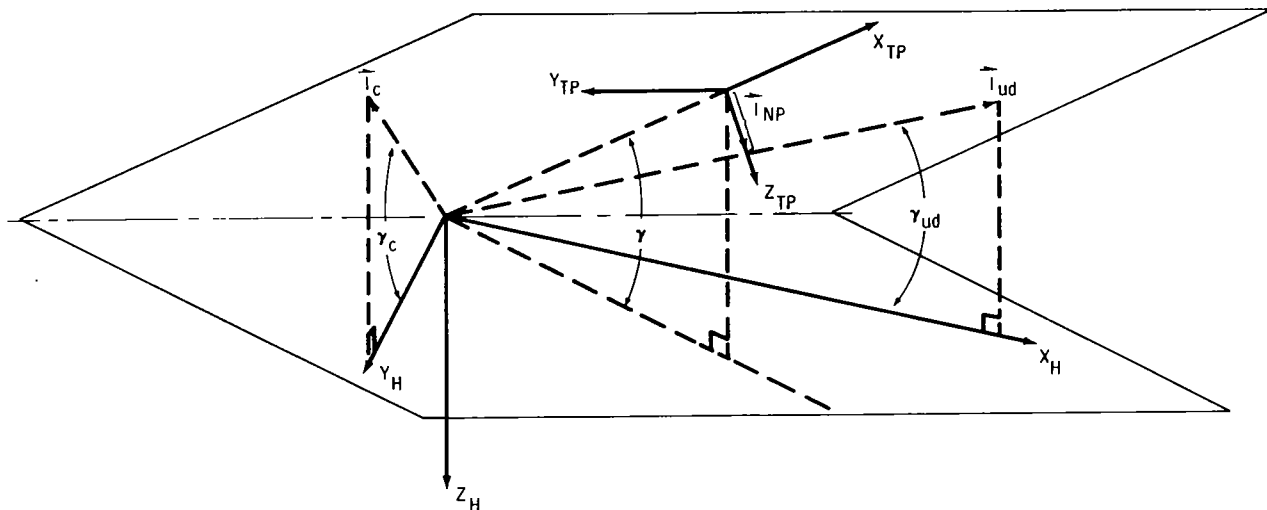

Figure A-1. - Local horizontal and tangent plane axis system.

DERIVATION OF THE RESULTANT WAVE SLOPE

The unit vectors referred to in the horizontal coordinate system in the upwind-downwind and crosswind directions $\vec{1}_{ud}$ and $\vec{1}_c$, respectively (shown in fig. A-1), can

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be calculated from γ_{ud} and γ_c. These unit vectors lie in the wave tangent plane and the vector cross product defines the normal to the wave tangent plane. The direction cosines for the unit vector normal to the wave tangent plane are

$$l_{NP} : m_{NP} : n_{NP} = \begin{vmatrix} 0 & -\sin \gamma_{ud} \\ \cos \gamma_c & -\sin \gamma_c \end{vmatrix} : \begin{vmatrix} -\sin \gamma_{ud} & \cos \gamma_{ud} \\ -\sin \gamma_c & 0 \end{vmatrix} : \begin{vmatrix} \cos \gamma_{ud} & 0 \\ 0 & \cos \gamma_c \end{vmatrix} \tag{A1}$$

or

$$\vec{1}_{NP} = l_{NP}\vec{i} + m_{NP}\vec{j} + n_{NP}\vec{k} \tag{A2}$$

In the wave tangent plane coordinate system X_{TP}, Y_{TP}, Z_{TP}, the $\vec{1}_{NP}$ is co-linear with Z_{TP}. Figure A-2 represents the relationship of X_{TP} and Z_{TP} to the desired wave angle γ. Therefore, the wave angle γ may be defined by the Z component of $\vec{1}_{NP}$ and the resultant of the X and Y components of $\vec{1}_{NP}$.
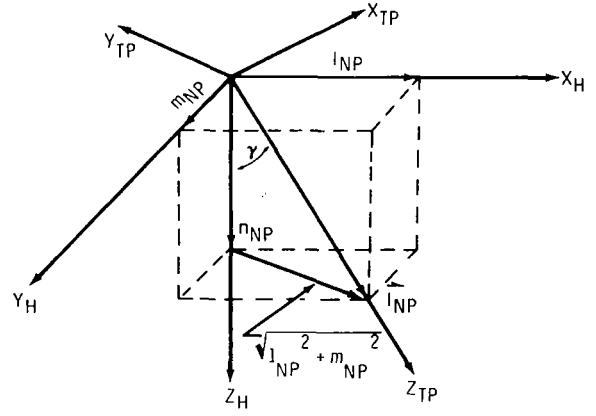

$$\gamma = \tan^{-1} \frac{\sqrt{l_{NP}^2 + m_{NP}^2}}{n_{NP}} \tag{A3}$$

Figure A-2. - Relationship of γ and X_H-Y_H plane.

which may be written in terms of the component slopes as

$$\gamma = \tan^{-1} \frac{\sqrt{\cos^2\gamma_c \sin^2\gamma_{ud} + \sin^2\gamma_c \cos^2\gamma_{ud}}}{\cos \gamma_{ud} \cos \gamma_c} \tag{A4}$$

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and finally

$$\gamma = \tan^{-1}\sqrt{\tan^2\gamma_{ud} + \tan^2\gamma_c} \qquad (A5)$$

STANDARD DEVIATION

To derive the equations for the standard deviation

$$\gamma_{ud}' = 0.79\left(0.808 \times 10^{-3}V - 0.00581\right)^{1/2} \text{ radians} \qquad (A6)$$

and

$$\gamma_c' = 0.612\left(0.808 \times 10^{-3}V - 0.00581\right)^{1/2} \text{ radians} \qquad (A7)$$

It must be remembered that the standard deviation equals the positive square root of the variance, or

$$\gamma' = +\sqrt{\sigma^2} \qquad (A8)$$

Therefore, the variance

$$\sigma^2 = \frac{1}{2}\int_{g/2v}^{2\pi/T} \frac{\pi}{2} \frac{C e^{-2g^2/\mu^2 v^2}}{\mu^2 g^2}\, d\mu \qquad (A9)$$

must be evaluated. To do so, equation (A9) must be written in general form. But first, all constants must be removed from within the integral.

$$\sigma^2 = \frac{\pi C}{4g^2}\int_{g/2v}^{2\pi/T} \frac{1}{\mu^2} e^{-2g^2/\mu^2 v^2}\, d\mu \qquad (A10)$$

Now, the equation can be written in general form.

$$\sigma^2 = A \int_\epsilon^\delta \frac{1}{\mu^2} e^{-B/\mu^2} d\mu \qquad (A11)$$

where $A = \pi C/4g^2$ and $B = 2g^2/v^2$. Let

$$\frac{B}{\mu^2} = \frac{w^2}{2} \qquad (A12)$$

Then

$$w^2 = \frac{2B}{\mu^2} \qquad (A13)$$

and

$$w = \frac{\sqrt{2B}}{\mu} \qquad (A14)$$

Then

$$\mu = \frac{\sqrt{2B}}{w} \qquad (A15)$$

and

$$d\mu = \frac{-\sqrt{2B}}{w^2} dw \qquad (A16)$$

The limits are

$$\mu_{upper} = \delta \qquad (A17)$$

and

$$\mu_{lower} = \epsilon \qquad (A18)$$

Now, solve

$$w = \frac{\sqrt{2B}}{\mu} \qquad (A19)$$

for w_{upper} and w_{lower}. By substitution

$$w_{upper} = \frac{\sqrt{2B}}{\mu_{upper}} = \frac{\sqrt{2\left(\frac{2g^2}{v^2}\right)}}{\frac{2\pi}{T}} = \frac{Tg}{\pi v} \qquad (A20)$$

and

$$w_{lower} = \frac{\sqrt{2B}}{\mu_{lower}} = \frac{\sqrt{2\left(\frac{2g^2}{v^2}\right)}}{\frac{g}{2v}} = 4 \qquad (A21)$$

Now, the equation can be written in standard form by following the substitutions that are indicated.

$$\sigma^2 = A \int_4^{Tg/\pi v} \frac{w^2}{2B} e^{-w^2/2} \left(-\frac{\sqrt{2B}}{w^2}\right) dw \qquad (A22)$$

$$\sigma^2 = -A \int_4^{Tg/\pi v} \frac{1}{\sqrt{2B}} e^{-w^2/2} dw \qquad (A23)$$

At this point, the equation must be multiplied by a factor equal to one, so that the tables for a normal curve may be used,

$$\sigma^2 = -A \left(\frac{\sqrt{2\pi}}{\sqrt{2\pi}} \right) \int_4^{Tg/\pi v} \frac{1}{\sqrt{2B}} e^{-w^2/2} dw \qquad (A24)$$

or,

$$\sigma^2 = \frac{A\sqrt{2\pi}}{\sqrt{2B}} \int_{Tg/\pi v}^4 \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw \qquad (A25)$$

The equation can be further simplified as follows. Knowing that $C = 3.05 \times 10^4 \text{ cm}^2/\text{sec}^5 = 3.05 \text{ m}^2/\text{sec}^5$ and $g = 980 \text{ cm/sec}^2 = 9.8 \text{ m/sec}^2$, these values can be substituted in the expressions for A and B.

$$A = 0.0249 \text{ sec}^{-1} \qquad (A26)$$

$$B = \frac{192.1}{v^2} \text{ sec}^{-2} \qquad (A27)$$

The equation becomes

$$\sigma^2 = 3.2 \times 10^{-3} v \int_{Tg/\pi v}^4 \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw \text{ radians} \qquad (A28)$$

Now, the equation can be solved for any value of velocity v in SI units by looking up the limits in the tables for normal curves. For example, solve for a v of 15 m/sec. The lower limit becomes $T(9.8)/15\pi$.

This discussion has been restricted to waves of a period of 1.6 seconds or greater. The lower limit is, then,

$$\frac{(1.6)9.8}{15\pi} = \frac{15.68}{47.124} = 0.333 \qquad (A29)$$

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The equation would now appear as

$$\sigma^2 = 48 \times 10^{-3} \int_{0.333}^{4.0} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw \qquad (A30)$$

From the tables

$$\sigma^2 = \left(48 \times 10^{-3}\right)(0.5000 - 0.1307) \qquad (A31)$$

$$\sigma^2 = \left(48 \times 10^{-3}\right)(0.3693) \qquad (A32)$$

$$\sigma^2 = 17.726 \times 10^{-3} \text{ radians} \qquad (A33)$$

For a velocity of 10 m/sec

$$\sigma^2 = \left(3.2 \times 10^{-3}\right)(10) \int_{0.499}^{4.0} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw \qquad (A34)$$

$$\sigma^2 = \left(32 \times 10^{-3}\right)(0.5000 - 0.1915) \qquad (A35)$$

$$\sigma^2 = \left(32 \times 10^{-3}\right)(0.3085) \qquad (A36)$$

$$\sigma^2 = 9.872 \times 10^{-3} \text{ radians} \qquad (A37)$$

Now, two values of σ^2 can be plotted as a function of velocity (in knots) to calculate the equation of the line drawn through the two points. First, meters per second is converted to knots for both velocities, which are as follows.

$$10 \text{ m/sec} = 19.426 \text{ knots} \qquad (A38)$$

$$15 \text{ m/sec} = 29.139 \text{ knots} \qquad (A39)$$

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The two values of σ^2 are plotted as a function of velocity (in knots) as shown in figure A-3. To calculate the equation of the line, the standard slope-intercept method is used.

$$y = mx + b \tag{A40}$$

where $b = -5.81 \times 10^{-3}$.

$$m = \frac{(17.73 + 5.81) \times 10^{-3}}{29} = 0.808 \times 10^{-3} \text{ rad/knots} \tag{A41}$$

The equation of the line is

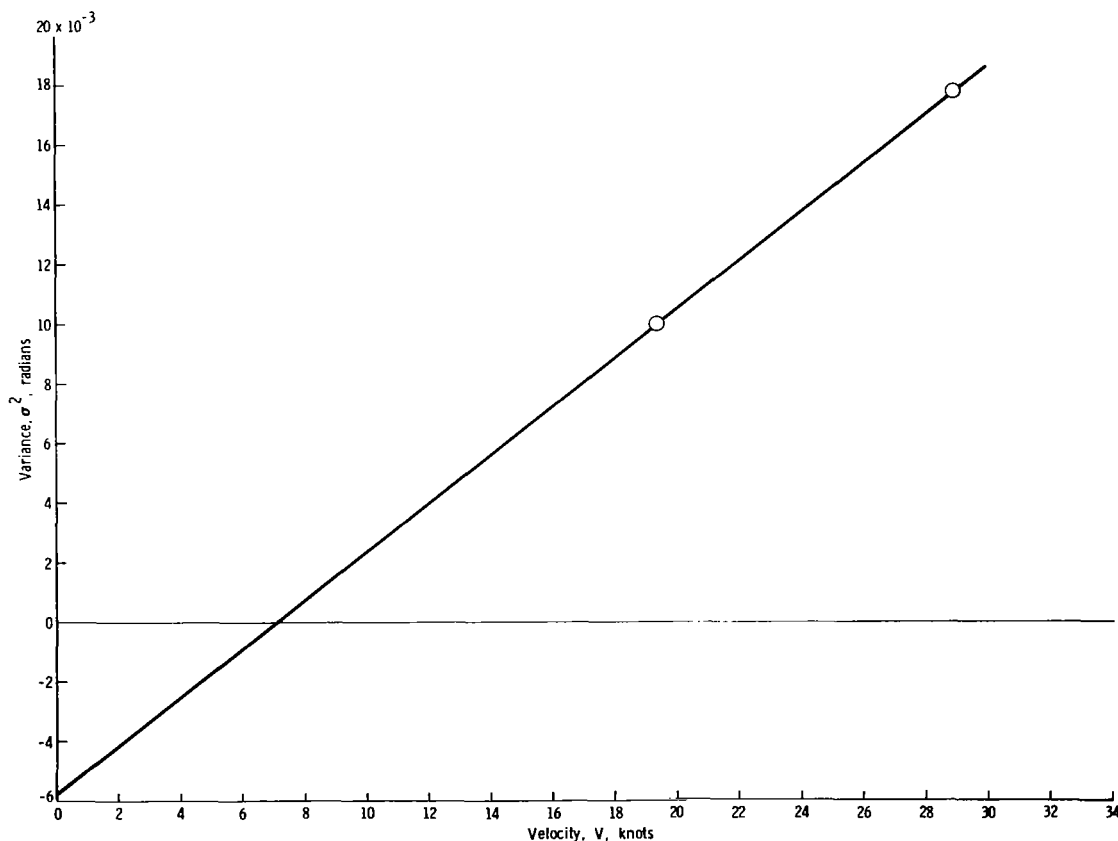$$y = \left(0.808 \times 10^{-3} V\right) - 0.00581 \text{ rad/knots} \tag{A42}$$

Figure A-3. - Correlation of variance and velocity.

The standard deviation equals the positive square root of the variance; therefore

$$\gamma' = +\sqrt{\sigma^2} = \left(0.808 \times 10^{-3}V - 0.00581\right)^{1/2} \text{ radians} \tag{A43}$$

For the upwind-downwind component

$$\gamma_{ud}' = \left[0.625 \left(0.808 \times 10^{-3}V - 0.00581\right)\right]^{1/2} \text{ radians}$$

$$= 0.79 \left(0.808 \times 10^{-3}V - 0.00581\right)^{1/2} \text{ radians} \tag{A44}$$

Likewise, the crosswind component becomes

$$\gamma_c' = 0.612 \left(0.808 \times 10^{-3}V - 0.00581\right)^{1/2} \text{ radians} \tag{A45}$$